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We study the collision of two slowly rotating, initially non boosted, black holes in the close limit. A "punctures" modification of the Bowen - York method is used to construct conformally flat initial data appropriate to the problem. We keep only the lowest nontrivial orders capable of giving rise to radiation of both gravitational energy and angular momentum. We show that even with these simplifications an extension to higher orders of the linear Regge-Wheeler-Zerilli black hole perturbation theory, is required to deal with the evolution equations of the leading contributing multipoles. This extension is derived, together with appropriate extensions of the Regge-Wheeler and Zerilli equations. The data is numerically evolved using these equations, to obtain the asymptotic gravitational wave forms and amplitudes. Expressions for the radiated gravitational energy and angular momentum are derived and used together with the results of the numerical evolution to provide quantitative expressions for the relative contribution of different terms, and their significance is analyzed.

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I. INTRODUCTION

One of the most promising sources of gravitational radiation currently under investigation is the collision and coalescence of binary pairs of black holes of comparable mass [1]. The dynamics of such systems in the regime corresponding to large separation, where the interaction is weak, appears to be adequately described in terms of post-Newtonian approximations [2]. At sufficiently close separation, where the interaction becomes strong, the system is expected to go through a "plunge" phase, eventually leading to the formation of a common horizon, after which the system behaves as a perturbed black hole that settles to a final stationary state. At this latest stage, "close approximation" perturbative method have proved capable of providing reliable information on waveforms and related quantities [3]. However, up to the present, no equally successful methods have been found that can handle the intermediate strong field regime, although some speculations have been presented, based mainly on extrapolations between the early and very late regimes. The fact that these methods may not reveal effects that may be unique to the strong field stage of binary interactions has been recently pointed out by Price and Whelan [4], in relation with the effect of tidal interactions in binary black hole inspiral, in the case where the black holes have moderately to large angular momentum.

One of the particular close limit regimes considered in [4], that in which the black holes have equal, parallel spins, pointing in the same direction and perpendicular to the line joining the centers of the black holes, had not been analyzed in detail previously in the literature [5], and therefore only rough estimates are given there. In this paper we provide that analysis by considering the collision of two slowly rotating, initially non boosted, black holes in the close limit. The black holes have equal masses and equal parallel spins aligned perpendicular to the line joining initially the centers of the black holes. The expression "initially non boosted" refers to the manner in which the initial data is constructed. In our case we consider initial data obtained by the "puncture" [6] modification of the Bowen-York ansatz [7]. Mostly for simplicity, we include angular, but not linear, momentum in the conformal extrinsic curvature. In the close limit considered here, this represents a perturbed, non axially symmetric, state of a (slowly) rotating black hole, and therefore it is natural to apply the Regge-Wheeler-Zerilli [8], [9] black hole perturbation theory to analyze its evolution. However, as we show in this paper, even with the above simplifications the evolution equations for a part of the leading contributions are linear, but not homogenous, and require formulating the problem in second order perturbation theory [10], for a problem with more than one perturbation parameter. The corresponding evolution equations contain "source" terms, and their influence on the asymptotic behavior of the solutions, which contains the information on the gravitational waves, needs careful evaluation. Since these expressions had not been given in the literature before, this paper contains the details of construction of the initial data and the evolution equations, together with the corresponding expressions for the gravitational wave amplitudes, and radiated energy, in terms of (gauge invariant) Regge-Wheeler-Zerilli functions.

In the close limit the system analyzed here bears a marked resemblance with the "pseudoinspiral problem" considered in [11], [12], where the binary black hole system has orbital, but no spin angular momentum. This resemblance stems from the fact that in the close limit, once a single horizon has been formed, both systems behave as distorted black hole settling to a final Kerr form, with the distortion dominated by quadrupolar terms. Since the systems lack axial symmetry, it is expected in both cases that the gravitational waves will carry also radiation of angular momentum. This expectation was confirmed by the computations carried out in [11], [12] for the "pseudoinspiral case". Since only linear perturbation theory is required in that case, the computations were done both in the Regge-Wheeler-Zerilli and the Teukolsky formalism [13]. However, it was rather surprisingly found that although the radiated energy computed in both formalism showed a reasonable agreement, this did not happen for the predicted radiation of angular momentum. In fact, except for very small values of the angular momentum, the Teukolsky formalism, as applied in [11], [12], leads to the rather disturbing result that the angular momentum increases as a result of the radiation (the angular momentum radiated has opposite sign to that of the system). This is not the case for the prediction given by the Regge-Wheeler formalism. Although the detailed reasons for this discrepancy are not clearly understood, it is the opinion of the present authors that it may be due mainly to the rather large departure of the initial data from that of a Kerr black hole. Since the system considered in this paper has initial data that is closer to that of a Kerr black hole, in the sense that the perturbations are of second order in the angular momentum parameter, rather than first order as in the case of [11], [12], one of the main purposes of our analysis was to compute the angular momentum radiated in the process, as the disturbed black hole settles to its final Kerr form, and its eventual comparison with the corresponding Teukolsky results. As is shown in this paper, within the Regge-Wheeler-Zerilli formalism, this, as well as the computation of the leading waveforms and radiated energy, requires an extension of the theory to second order, (in fact, to even higher order, if we derive the expression for the radiated angular momentum as indicated in Section VII), which is explicitly given here. It should be clear that at this point that the alternative treatment be based on the Teukolsky formalism, and, possibly, its extension to second order [14], is a task requiring as much extension as the one reported here. For this reason, that development, and its comparison with the results obtained with the Regge-Wheeler formalism presented here, will be considered in a separate paper, currently under development.

The paper is organized as follows. In the next Section we consider the construction of the initial data. This is naturally given in a "Misner" type gauge. However, as we indicate, this gauge is not appropriate for a perturbative computation, and we derive the gauge transformation to, and the form of the initial data in a "Boyer-Lindquist-Kerr" type of gauge for the $\ell = 1$, odd parity perturbations, where these have a very simple coordinate dependence. We then consider the evolution of the second order perturbations in the Regge-Wheeler gauge, and obtain the corresponding Zerilli, and Regge-Wheeler functions and equations. We describe their asymptotic behavior, their relation to the corresponding gravitational wave amplitudes, and derive the expressions for the radiated power and energy. Finally, we obtain an expression for the radiated angular momentum directly using perturbation theory, in terms of the Zerilli functions for the perturbations. Comments and conclusions are given in the last Section.

II. INITIAL DATA

In applications of the Regge - Wheeler - Zerilli black hole perturbation theory one considers a family of metrics, depending on one or a few parameters, such that the Schwarzschild metric is recovered when the parameters are set equal to zero. The problem may be analyzed as an initial value problem, so that the specification of the members of the family amounts to identifying the corresponding initial data on a certain three dimensional hypersurface. Actually, since the computations are carried out in a coordinate patch that covers only the region outside the black hole horizon, one requires expansions of the initial that may be convergent only in that region. Examples are furnished by the families of initial data constructed via the "conformal approach" [7] to the initial value problem in General Relativity. In this approach one assumes that the three-metric g_{ab} is conformally flat, i.e. $g_{ab} = \Phi^4 \eta_{ab}$, where Φ is some non vanishing function, and η_{ab} a flat three-metric, and defines a conformal extrinsic curvature $\widehat{K}_{ab} = \Phi^2 K_{ab}$. Then, assuming maximal slicing $K^{ab} g_{ab} = 0$, the initial value constraint (for a vacuum metric) are reduced to,

$$\nabla_a \widehat{K}^{ab} = 0 \tag{1}$$

$$\nabla^2 \Phi = -\frac{1}{8} \frac{\widehat{K}^{ab} \widehat{K}_{ab}}{\Phi^7} \tag{2}$$

where all derivatives are with respect to the flat metric. In general, the conditions,

$$\Phi > 0 \ , \quad \lim_{\rho\to\infty} \Phi = 1. \tag{3}$$

where ρ is a radial coordinate on the fiducial flat space where η is defined, are imposed on Φ. Therefore, on the initial slice, asymptotically, for large ρ we have $\widehat{K}^{ab} \to K^{ab}$, and $g_{ab} \to \eta_{ab}$. This fact is important because one can find exact solutions to the momentum constraint (1), and, using (3), obtain a (partial) physical interpretation of the associated spacetime in terms of ADM observables, without solving (2) explicitly. For example, in [15], the solution

$$\widehat{K}_{ab} = \frac{3}{\rho^3}\left[\epsilon_{acd}S^c n^d n_b + n_a \epsilon_{bcd} S^c n^d\right], \tag{4}$$

where $\rho = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, $n^a = x^a/\rho$, with x^a cartesian coordinates on the flat space background, and $\mathbf{S}$ is a constant vector, is considered. Then, taking spherical polar coordinates for the flat background, so that

$$\eta_{ab}dx^a dx^b = (d\rho)^2 + \rho^2(d\theta)^2 + \rho^2\sin(\theta)^2(d\phi)^2 \tag{5}$$

one straightforwardly identifies S^a with the angular momentum of the system, irrespective of the detailed knowledge of Φ. In the application given in [15], the conditions (3) are supplemented with appropriate boundary conditions for Φ, so that the resulting initial data is identified with that of a single spinning black hole (in a non stationary initial state), centered at the origin of the flat coordinate system.

In this paper we consider the initial data that results from taking $\widehat{K}_{ab}$ as the sum of two terms of the form (4), symmetrically centered around the coordinate origin, with the same S^a. In the cartesian coordinate system associated with the flat background, we have,

$$\begin{aligned}\widehat{K}_{ab} = &\frac{3}{\rho_1^3}\left[\epsilon_{acd}S^c n_1^d n_{1\ b} + n_{1\ a}\epsilon_{bcd}S^c n_1^d\right] \\ &+\frac{3}{\rho_2^3}\left[\epsilon_{acd}S^c n_2^d n_{2\ b} + n_{2\ a}\epsilon_{bcd}S^c n_2^d\right]\end{aligned} \tag{6}$$

where $\rho_1 = \sqrt{(x^1 - L/2)^2 + (x^2)^2 + (x^3)^2}$, $\rho_2 = \sqrt{(x^1 + L/2)^2 + (x^2)^2 + (x^3)^2}$, $n_1^a = x^a/\rho_1$, and $n_2^a = x^a/\rho_2$. We also take $S^a = (0, 0, S)$. This may be thought of as the (conformal) extrinsic curvature corresponding to two black holes with equal angular momentum S, pointing along the z-axis, placed at the points $(\pm L/2, 0, 0)$. Actually, this interpretation requires that Φ has a singularity structure in accordance with that of (6). This may be achieved in different ways. Here we take the "punctures" ansatz [6], namely, we assume that Φ is of the form,

$$\Phi = \Phi_{BL} + \Phi_{Reg} \tag{7}$$

where,

$$\Phi_{BL} = 1 + \frac{M_0}{2\rho_1} + \frac{M_0}{2\rho_2} \tag{8}$$

that is, Φ_{BL} is taken as the Brill-Lindquist conformal factor [16], and we impose that Φ_{Reg} must be regular in the whole conformal plane, and vanish for large ρ_i. In accordance with this prescription, Φ_{Reg} satisfies the equation,

$$\nabla^2\Phi_{Reg} = -\frac{1}{8}\frac{\widehat{K}^{ab}\widehat{K}_{ab}}{(\Phi_{BL} + \Phi_{Reg})^7} \tag{9}$$

with $\widehat{K}^{ab}$ given by (6), which, for $S \neq 0$, may only be solved numerically. We notice, however, that for $S = 0$ we should have $\Phi_{Reg} = 0$, and, from the form (9), we expect that Φ_{Reg} admits an expansion in powers of S^2, near $S = 0$. Moreover, when $L \to 0$, Φ_{BL} approaches the conformal factor for a Schwarzschild black hole. Therefore, the ansatz (6), together with (7), and (8), lead to a two-parameter family of initial data, with the Schwarzschild metric as the limit when the parameters vanish. In the slow, close approximation approach of this paper we consider the situation when the two holes are initially so close to each other that a single horizon is formed, and the system, from the point of view external to the horizon, may be considered as a perturbed single black hole. In this situation, the external field admits a multipolar expansion, and the multipolar order turns out to be correlated with different powers of the parameters, so that, by keeping the lowest orders in the parameters, we end up with a few multipolar terms, and this is expected to provide a good approximation to the exact initial data, and its evolution.

There is always a certain degree of arbitrariness in the choice of perturbation parameters. Here we notice that, because of the conditions imposed on Φ, the initial data corresponds to a system with ADM angular momentum $J = 2S$. We shall therefore consider J, and L as expansion parameters. In the limit $J = 0$, $L = 0$, the conformal factor Φ corresponds to that of a Schwarzschild black hole of mass $M = 2M_0$. In what follows we write all expressions in terms of M, rather than M_0.

A. Perturbative expansion of the initial data

We recall that in the Regge-Wheeler black hole perturbation theory the analysis is restricted to the region outside the horizon. To construct our perturbative initial data we start by considering (9), but restricted to the region $\rho > L$. If we expand $\widehat{K}_{ab}$ in powers of L/ρ, and keep only lowest orders, we find,

$$
\begin{aligned}
\widehat{K}_{\rho\rho} &= -3\frac{JL^2\sin(2\phi)\sin^2(\theta)}{\rho^5} \\
\widehat{K}_{\rho\theta} &= -\frac{9}{8}\frac{JL^2\sin(2\phi)\sin(\theta)\cos(\theta)}{\rho^4} \\
\widehat{K}_{\rho\phi} &= 3\frac{J\sin^2(\theta)}{\rho^2} - \frac{3}{8}\frac{JL^2(7+\cos^2(\phi)(25\cos^2(\theta)-19))\sin^2(\theta)}{\rho^4} \\
\widehat{K}_{\theta\theta} &= \frac{3}{4}JL^2\sin(2\phi)\cos^2(\theta)/\rho^3 \\
\widehat{K}_{\theta\phi} &= -\frac{3}{4}\frac{JL^2\sin(\theta)\cos(\theta)(1+3\cos^2(\phi)-5\cos^2(\phi)\cos^2(\theta)}{\rho^3} \\
\widehat{K}_{\phi\phi} &= \frac{3}{4}\frac{JL^2\sin(2\phi)(4-9\cos^2(\theta)+5\cos^4(\theta)}{\rho^3}
\end{aligned}
\tag{10}
$$

This implies, keeping again only the lowest contributing order,

$$
\widehat{K}_{ab}\widehat{K}^{ab} = 18J^2\frac{\sin^2(\theta)}{\rho^6} \tag{11}
$$

Similarly, we expand Φ_{BL},

$$
\Phi_{BL} = 1 + \frac{M}{2\rho} - \frac{M(1-3\cos^2(\phi)\sin^2(\theta))L^2}{16\rho^3} \tag{12}
$$

A simple power counting shows that the lowest order in Φ_{Reg} is J^2, if we consider J^2 and L^2 of the same order. To this order (9) reduces to,

$$
\nabla^2\Phi_{Reg} = -288J^2\frac{\sin^2(\theta)\rho}{(2\rho+M)^7} \tag{13}
$$

The solution of this equation is,

$$
\Phi_{Reg} = \Phi_{(0,0)}(\rho)Y_0^0(\theta,\phi) + \Phi_{(2,0)}(\rho)Y_2^0(\theta,\phi) \tag{14}
$$

where $Y_\ell^m(\theta,\phi)$ is a standard spherical harmonic, and,

$$
\begin{aligned}
\Phi_{(0,0)}(\rho) &= \frac{4\sqrt{\pi}J^2(M^4+10\rho M^3+40\rho^2M^2+40\rho^3M+16\rho^4)}{5M^3(2\rho+M)^5} + C_1 + \frac{C_2}{\rho} \\
\Phi_{(2,0)}(\rho) &= \frac{\sqrt{5\pi}J^2(M^4+10\rho M^3+40\rho^2M^2+80\rho^3M+80\rho^4)}{25\rho^3(2\rho+M)^5} + C_3\rho^2 + \frac{C_4}{\rho^3}
\end{aligned}
\tag{15}
$$

We must set $C_1 = C_3 = 0$, to have the correct asymptotic behavior. The constants C_2, and C_4 are fixed so that $\Phi_{(0,0)}$, and $\Phi_{(2,0)}$ are regular for $\rho = 0$. The final result is,

$$
\begin{aligned}
\Phi_{(0,0)}(\rho) &= \frac{4\sqrt{\pi}J^2(M^4+10\rho M^3+40\rho^2M^2+40\rho^3M+16\rho^4)}{5M^3(2\rho+M)^5} \\
\Phi_{(2,0)}(\rho) &= -\frac{32\sqrt{5\pi}J^2\rho^2}{25M(2\rho+M)^5}
\end{aligned}
\tag{16}
$$

This completes our perturbative computation of the conformal factor and extrinsic curvature. To construct the initial data for the evolution equations, we recall that if we fix the shift functions $N_a = 0$, we have

$$
\begin{aligned}
ds^2 &= g_{ab}dx^a dx^b - N^2 dt^2 \\
\frac{\partial g_{ab}}{\partial t} &= -2NK_{ab}
\end{aligned}
\tag{17}
$$

where for $t = 0$ we identify g_{ab}, and K_{ab} with those obtained above. We next change from the conformal radial coordinate ρ to a Schwarzschild radial coordinate r, such that $\rho = (\sqrt{r}+\sqrt{r-2M})^2/4$, and choose $N = \sqrt{1-2M/r}$, so that we recover the Schwarzschild metric, with mass M, for $J = L = 0$. The final results, in the standard Regge-Wheeler multipolar decomposition and notation [8] (see. e.g. [10], for more details) are listed below.

B. Metric components

The metric components obtained using the procedure just described may be written as follows,

1. $\ell = 0$

$$
H2_{0,\,0} = K_{0,\,0} = \frac{4}{5}\,\frac{\sqrt{\pi}\,J^2\,(r\,M + M^2 + 2\,r^2)}{r^3\,M^3}
\tag{18}
$$

2. $\ell = 2$

$$
H2_{2,\,0} = K_{2,\,0} = -\frac{4}{25}\,\frac{\sqrt{5}\,\sqrt{\pi}\,J^2}{M\,r^3} - \frac{8}{5}\,\frac{M\,\sqrt{5}\,\sqrt{\pi}\,L^2}{\sqrt{r}\,(\sqrt{r}+\sqrt{r-2\,M})^5}
\tag{19}
$$

$$
H2_{2,\,-2} = H2_{2,\,2} = \frac{4}{5}\,\frac{\sqrt{\pi}\,L^2\,M\,\sqrt{30}}{\sqrt{r}\,(\sqrt{r}+\sqrt{r-2\,M})^5}
\tag{20}
$$

$$
K_{2,\,-2} = K_{2,\,2} = \frac{4}{5}\,\frac{\sqrt{\pi}\,L^2\,M\,\sqrt{30}}{\sqrt{r}\,(\sqrt{r}+\sqrt{r-2\,M})^5}
\tag{21}
$$

$$
\partial_t H2_{2,\,-2} = -\partial_t H2_{2,\,2} = \frac{64}{5}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{r-2\,M}\,\sqrt{30}}{r^{(7/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^4}
\tag{22}
$$

$$
\partial_t h1_{2,\,-2} = -\frac{8}{15}\,\frac{I\,J\,(8\,r + 8\,\sqrt{r}\,\sqrt{r-2\,M} - 3\,M)\,\sqrt{\pi}\,L^2\,\sqrt{30}}{r^{(5/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^5}
\tag{23}
$$

$$
\partial_t h1_{2,\,2} = -\partial_t h1_{2,\,-2}
\tag{24}
$$

$$
\partial_t G_{2,\,-2} = -\partial_t G_{2,\,2} = \frac{16}{15}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{r-2\,M}\,\sqrt{30}}{r^{(7/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^4}
\tag{25}
$$

$$
\partial_t K_{2,\,-2} = -\partial_t K_{2,\,2} = -\frac{16}{5}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{r-2\,M}\,\sqrt{30}}{r^{(7/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^4}
\tag{26}
$$

3. $\ell = 1$, *odd*

$$
\partial_t k1_{1,\,0} = 4\,\frac{\sqrt{3}\,J\,\sqrt{\pi}}{r^2} - \frac{16}{5}\,\frac{\sqrt{3}\,J\,\sqrt{\pi}\,L^2\,M}{(\sqrt{r}+\sqrt{r-2\,M})^5\,r^{(5/2)}}
\tag{27}
$$

4. $\ell = 3$, odd

$$\partial_t k1_{3,-2} = \frac{4}{105}\,\frac{J\,L^2\,\sqrt{\pi}\,\sqrt{210}(25\,r + 25\,\sqrt{r}\,\sqrt{r-2\,M} - 6\,M)}{r^{(5/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^5} \tag{28}$$

$$\partial_t k1_{3,0} = -\frac{8}{35}\,\frac{\sqrt{\pi}\,\sqrt{7}\,L^2\,J\,(25\,r + 25\,\sqrt{r}\,\sqrt{r-2\,M} - 6\,M)}{(\sqrt{r}+\sqrt{r-2\,M})^5\,r^{(5/2)}} \tag{29}$$

$$\partial_t k1_{3,2} = \frac{4}{105}\,\frac{J\,L^2\sqrt{\pi}\,\sqrt{210}\,(25\,r + 25\,\sqrt{r}\,\sqrt{r-2\,M} - 6\,M)}{r^{(5/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^5} \tag{30}$$

$$\partial_t k2_{3,-2} = \partial_t k2_{3,2} = \frac{8}{21}\,\frac{J\,\sqrt{\pi}\,L^2\,\sqrt{r-2\,M}\,\sqrt{210}}{r^{(3/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^4} \tag{31}$$

$$\partial_t k2_{3,0} = -\frac{16}{7}\,\frac{\sqrt{\pi}\,\sqrt{r-2\,M}\,L^2\,J\,\sqrt{7}}{(\sqrt{r}+\sqrt{r-2\,M})^4\,r^{(3/2)}} \tag{32}$$

C. Extrinsic curvature components

The extrinsic curvature components may be written in the form,

1. $\ell = 1$,odd

$$Kk1_{1,0} = -4\,\frac{\sqrt{3}\,J\,\sqrt{\pi}}{r^{(3/2)}\,\sqrt{r-2\,M}} + \frac{16}{5}\,\frac{\sqrt{3}\,J\,\sqrt{\pi}\,L^2\,M}{(\sqrt{r}+\sqrt{r-2\,M})^5\,r^2\,\sqrt{r-2\,M}} \tag{33}$$

2. $\ell = 2$, even

$$KH2_{2,-2} = -KH2_{2,2} = -\frac{32}{5}\,\frac{I\,L^2\,J\,\sqrt{30}\,\sqrt{\pi}}{(\sqrt{r}+\sqrt{r-2\,M})^4\,r^2\,(r-2\,M)} \tag{34}$$

$$Kh1_{2,-2} = -Kh1_{2,2} = \frac{4}{15}\,\frac{I\,J\,(8\,r + 8\,\sqrt{r}\,\sqrt{r-2\,M} - 3\,M)\,\sqrt{\pi}\,L^2\,\sqrt{30}}{r^2\,\sqrt{r-2\,M}\,(\sqrt{r}+\sqrt{r-2\,M})^5} \tag{35}$$

$$KG_{2,-2} = -KG_{2,2} = -\frac{8}{15}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{30}}{r^3\,(\sqrt{r}+\sqrt{r-2\,M})^4} \tag{36}$$

$$KK_{2,-2} = -KK_{2,2} = \frac{8}{5}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{30}}{r^3\,(\sqrt{r}+\sqrt{r-2\,M})^4} \tag{37}$$

3. $\ell = 3$, odd

$$Kk1_{3,-2} = Kk1_{3,0} = -\frac{2}{105}\,\frac{J\,(25\,r + 25\,\sqrt{r}\,\sqrt{r-2\,M} - 6\,M)\,\sqrt{\pi}\,L^2\,\sqrt{210}}{r^2\,\sqrt{r-2\,M}\,(\sqrt{r}+\sqrt{r-2\,M})^5} \tag{38}$$

$$Kk1_{3,0} = \frac{4}{35}\,\frac{\sqrt{\pi}\,\sqrt{7}\,L^2\,J\,(25\,r + 25\,\sqrt{r}\,\sqrt{r-2\,M} - 6\,M)}{(\sqrt{r}+\sqrt{r-2\,M})^5\,r^2\,\sqrt{r-2\,M}} \tag{39}$$

$$Kk2_{3,-2} = Kk2_{3,2} = -\frac{4}{21}\,\frac{J\,\sqrt{\pi}\,L^2\,\sqrt{210}}{r\,(\sqrt{r}+\sqrt{r-2\,M})^4} \tag{40}$$

$$Kk2_{3,0} = \frac{8}{7}\,\frac{\sqrt{\pi}\,L^2\,J\,\sqrt{7}}{(\sqrt{r}+\sqrt{r-2\,M})^4\,r} \tag{41}$$

III. THE CHOICE OF GAUGE FOR THE $\ell = 1$, ODD PARITY, PERTURBATION

If we consider the initial data for the $\ell = 1$, odd, perturbation, we find that if we set $k0_{1,0} = 0$, consistent with setting the shifts to zero, the evolution equations have the solution,

$$k1_{1,\,0} = 4\,\frac{\sqrt{3}\,J\,\sqrt{\pi}t}{r^2} \tag{42}$$

This in turn leads to solutions for the other amplitudes that grow in time. For our calculations it will be more convenient to choose a gauge where,

$$k0_{1,0} = \frac{4\sqrt{\pi}J}{\sqrt{3}r} \tag{43}$$

$$k1_{1,0} = 0 \tag{44}$$

The gauge transformation required to produce this change is first order in J, for the $\ell = 1$, odd parity terms, and introduces changes in the remaining terms of order J^2, and JL^2. A simple computation, using second order perturbation theory, as given e.g. in [10], shows that for the $\ell = 2$, even parity, order J^2 terms, the initial data resulting from this change is the same as for the initial gauge except that now we have,

$$H0_{2,\,0} = \frac{16\sqrt{5\pi}J^2}{15r^3(r-2M)} \tag{45}$$

Similarly, the transformation induces changes in the $\ell = 2$, even, order JL^2 terms. In this case, these correspond to first order perturbation, because the terms are linear in J. A straightforward computation shows that the expressions derived previously are replaced by,

$$\partial_t H2_{2,\,-2} = -\partial_t H2_{2,\,2} = \frac{128}{5}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{r-2\,M}\,\sqrt{30}}{r^{(7/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^4} \tag{46}$$

$$\partial_t h1_{2,\,-2} = -\frac{16}{15}\,\frac{I\,J\,(8\,r+8\,\sqrt{r}\,\sqrt{r-2\,M}-3\,M)\,\sqrt{\pi}\,L^2\,\sqrt{30}}{r^{(5/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^5} \tag{47}$$

$$\partial_t h1_{2,\,2} = -\partial_t h1_{2,\,-2} \tag{48}$$

$$\partial_t G_{2,\,-2} = -\partial_t G_{2,\,2} = \frac{32}{15}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{r-2\,M}\,\sqrt{30}}{r^{(7/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^4} \tag{49}$$

$$\partial_t K_{2,\,-2} = -\partial_t K_{2,\,2} = -\frac{32}{5}\,\frac{I\,J\,\sqrt{\pi}\,L^2\,\sqrt{r-2\,M}\,\sqrt{30}}{r^{(7/2)}\,(\sqrt{r}+\sqrt{r-2\,M})^4} \tag{50}$$

where we now have a non vanishing shift term ($\partial_t h0_{2,\pm 2} \neq 0$).

Finally, the gauge transformation equations for the $\ell = 3$, $m = 0, \pm 2$, odd parity perturbations take the form,

$$\begin{aligned}
\widetilde{k0}_{3,0} &= \frac{2}{\sqrt{35}r}\left[KL_{2,0} - 2GL_{2,0}\right] \\
\widetilde{k1}_{3,0} &= k1_{3,0} - \frac{6}{\sqrt{35}r^2}\left[KL_{2,0} - 2GL_{2,0}\right] \\
\widetilde{k2}_{3,0} &= k2_{3,0} \\
\widetilde{k0}_{3,\pm 2} &= \frac{2\sqrt{7}}{21r}\left[KL_{2,\pm 2} - 2GL_{2,\pm 2}\right] \\
\widetilde{k1}_{3,\pm 2} &= k1_{3,\pm 2} - \frac{2}{\sqrt{7}r^2}\left[KL_{2,\pm 2} - 2GL_{2,\pm 2}\right] \\
\widetilde{k2}_{3,\pm 2} &= k2_{3,\pm 2}
\end{aligned} \tag{51}$$

These equations may be used to give explicit expressions for these terms as functions of r. We shall not display them here.

IV. THE EVOLUTION EQUATIONS FOR THE INITIAL DATA

The time evolution of the initial data we have just constructed may be handled as follows. First we notice that the initial data depends on two parameters: J,and L. Actually, only even powers of L appear. This implies that, in a Regge-Wheeler type of expansion, terms of order J, or of order L^2 satisfy linear perturbation equation. We will, therefore, consider J, and L^2 as our perturbation parameters, and will restrict to the leading, non trivial, contributing orders and multipoles. In our case they may be classified as follows. a) Terms of order L^2. These correspond to the $\ell = 2$, even parity, multipole, with $m = 0, \pm 2$, and satisfy linear perturbation equations. b) Terms of order J that correspond to the $\ell = 1$, odd, multipole with $m = 0$, and are related to the angular momentum. Their time evolution is known explicitly. c) Terms of order J^2, with $\ell = 2$, even parity, $m = 0$. These are second order in J, and obey linear equations with a "source", and may be handled as in [15], but here we shall use second order perturbation theory to analyze their evolution. d) Terms of order JL^2. These are actually higher order in perturbation theory. Since terms of order J have $\ell = 1$, and are of odd parity, and terms of order L^2 have $\ell = 2$, and are of even parity, in principle, they may contribute "sources" for terms of order JL^2, and $\ell = 1, 2, 3$, with both even and odd parity. However, an analysis of the form of the corresponding Einstein equations shows that the perturbations corresponding to $\ell = 1, 3$, with even parity, and $\ell = 2$ with odd parity, obey linear homogeneous equations. Since the initial data for these terms vanishes, they vanish for all times. Of the remaining terms, those with $\ell = 1$ odd parity do not contribute to the radiation, and therefore, shall not be considered in what follows. In the next subsections we obtain and display the relevant equation for the evolution of the terms contributing to the radiation of angular momentum and energy, up to the orders considered. This implies, in some cases, that we shall have to extend previous results on the application of black hole perturbation theory in the Regge-Wheeler-Zerilli framework. These will be indicated where appropriate.

A. Order L^2

In this case we have to consider terms of even parity with $\ell = 2$, and $m = 0, \pm 2$. The reality of the metric implies that the amplitudes with $m = -2$ are the complex conjugate of the corresponding amplitude with $m = +2$, so we need to consider only one of these. Since all these perturbations obey first order equations, in the Regge-Wheeler gauge they may be uniquely written in terms of the Zerilli functions $\psi_{2,m}(t, r)$, $m = 0, \pm 2$, in the form [10],

$$\begin{aligned}
H02_{2,m}(t, r) &= H22_{2,m}(t, r) \\
H22_{2,m}(t, r) &= (r - 2M)\frac{\partial^2 \psi_{2,m}(t, r)}{\partial r^2} + \frac{(2r^2 - 2rM + 3M^2)}{r(2r + 3M)}\frac{\partial \psi_{2,m}(t, r)}{\partial r} \\
&\quad -3\,\frac{(3M^3 + 6rM^2 + 4r^2M + 4r^3)}{(2r + 3M)^2 r^2}\psi_{2,m}(t, r) \\
K2_{2,m}(t, r) &= \frac{(6r^2 + 6rM + 6M^2)}{r^2(2r + 3M)}\psi_{2,m}(t, r) + (1 - 2\frac{M}{r})(\frac{\partial}{\partial r}\psi_{2,m}(t, r)) \\
H12_{2,m}(t, r) &= \frac{(2r^2 - 6rM - 3M^2)}{(r - 2M)(2r + 3M)}\frac{\partial \psi_{2,m}(t, r)}{\partial t} + r\,(\frac{\partial^2}{\partial t\,\partial r}\psi_{2,m}(t, r))
\end{aligned} \tag{52}$$

The Zerilli functions $\psi_{2,m}(t, r)$ are given in a general gauge by the Moncrief [17] expression,

$$\begin{aligned}
\psi_{2,m}(t, r) &= \frac{r(r - 2M)}{3(2r + 3M)}\left[H22_{2,m} - r\frac{\partial K2_{2,m}}{\partial r} - \frac{r - 3M}{r - 2M}K2_{2,m}\right] \\
&\quad + \frac{r^2}{(2r + 3M)}\left[K2_{2,m} + (r - 2M)\left(\frac{\partial G2_{2,m}}{\partial r} - \frac{2}{r}h1_{2,m}\right)\right]
\end{aligned} \tag{53}$$

and obey the equation,

$$\frac{\partial^2 \psi_{2,m}}{\partial t^2} = \frac{\partial^2 \psi_{2,m}}{\partial r^{*2}} - 6\frac{(r - 2M)(3M^3 + 6rM^2 + 4r^2M + 4r^3)}{r^4(2r + 3M)^2}\psi_{2,m} \tag{54}$$

where,

$$r^* = r + 2M\ln(r/(2M) - 1) \tag{55}$$

These functions are directly related to the gravitational wave amplitudes. The appropriate expressions are given below.

The initial data are,

$$\psi_{2,\,0}|_{t=0} = -\frac{4\sqrt{5\pi}r(7\sqrt{r}+5\sqrt{r-2M})M}{15(\sqrt{r}+\sqrt{r-2M})^5(2r+3M)}$$
$$\left.\frac{\partial\psi_{2,\,0}}{\partial t}\right|_{t=0} = 0 \tag{56}$$

and,

$$\psi_{2,\,\pm 2}|_{t=0} = \frac{2\sqrt{30\pi}r(7\sqrt{r}+5\sqrt{r-2M})M}{15(\sqrt{r}+\sqrt{r-2M})^5(2r+3M)}$$
$$\left.\frac{\partial\psi_{2,\,\pm 2}}{\partial t}\right|_{t=0} = 0 \tag{57}$$

and we notice that, since all the $\psi_{2,i}(t,r)$ satisfy the same equation this implies

$$\psi_{2,\pm 2}(t,r) = -\sqrt{\frac{3}{2}}\,\psi_{2,0}(t,r) \tag{58}$$

To compute the wave forms, and radiated angular momentum and energy, we need to know the asymptotic behavior of the Regge-Wheeler gauge functions. This is immediately obtained from the knowledge of the asymptotic behavior of the corresponding Zerilli functions. Proceeding as indicated in [10], we obtain,

$$\psi_{2,m}(t,r) = \psi^{(0)}_{2,m}(t-r^*) + \psi^{(1)}_{2,m}(t-r^*)/r + \psi^{(12}_{2,m}(t-r^*)/r^2 + \psi^{(3)}_{2,m}(t-r^*)/r^3 + \mathcal{O}(1/r^4) \tag{59}$$

where the functions $\psi^{(i)}_{2,m}(x)$ may be written uniquely in terms of a single function $\mathcal{F}_{2,m}$ in the form

$$\psi^{(0)}_{2,m}(x) = \frac{d^2\mathcal{F}_{2,m}}{dx^2} \quad , \quad \psi^{(1)}_{2,m}(x) = 3\frac{d\mathcal{F}_{2,m}}{dx}$$
$$\psi^{(2)}_{2,m}(x) = 3\mathcal{F}_{2,m} - 3M\frac{d\mathcal{F}_{2,m}}{dx} \quad , \quad \psi^{(3)}_{2,m}(x) = -3M\mathcal{F}_{2,m} + \frac{21M^2}{4}\frac{d\mathcal{F}_{2,m}}{dx} \tag{60}$$

where $m = 0, \pm 2$, and the functions $\mathcal{F}_{2,m}$ are determined by the initial data, through the evolution equations.

The relation between the functions $\psi_{2,m}$ and the gravitational wave amplitudes results from their gauge invariance. In an asymptotically flat gauge we find that, for large r, and t, we have

$$\psi_{2,m}(t,r) \simeq rG_{2,m}(t,r) \tag{61}$$

where $\simeq$ implies equality up to terms of order r^{-1}. This result is used in Sections VI and VII, to compute the radiation of energy and angular momentum.

B. Order J^2

To order J^2 we only have perturbations for $\ell = 0$, (that contribute only to the ADM mass), and with $\ell = 2$, $m = 0$, of even parity, that play a role in the wave forms and radiated energy. The relevant part of the evolution of these terms may be obtained in different forms. One way is that considered in [15]. Here we describe an alternative treatment. As indicated previously, we fix the first order (order J) gauge, to the "Boyer - Lindquist - Kerr" (BLK) form, $k1_{1,0}(t,r) = 0$, $k0_{1,0} = 4\sqrt{(\pi/3)}J/r$. This implies a second order (order J^2) gauge transformation on the original Bowen - York order J^2 data that, as indicated, introduces only a (time independent) change in the lapse function $H0$.

Writing the second order Einstein equations it is easy to prove that the Zerilli - Moncrief function defined by,

$$\chi^{J^2}_{2,\,0}(t,\,r) = \frac{r(r-2M)}{3(2r+3M)}\left[H2_{2,\,0} - r\frac{\partial K_{2,\,0}}{\partial r} - \frac{r-3M}{r-2M}K_{2,\,0}\right]$$
$$+\frac{r^2}{(2r+3M)}\left[K_{2,\,0} + (r-2M)\left(\frac{\partial G_{2,\,0}}{\partial r} - \frac{2}{r}h1_{2,\,0}\right)\right] \tag{62}$$

where $H2$, $h1$, K, and G are the corresponding Regge-Wheeler coefficients of order J^2, obeys the equation,

$$\frac{\partial^2 \chi^{J^2}_{2,0}}{\partial t^2} = \frac{\partial^2 \chi^{J^2}_{2,0}}{\partial r^{*2}} - 6\frac{(r-2M)(3M^3+6rM^2+4r^2M+4r^3)}{r^4(2r+3M)^2}\chi^{J^2}_{2,0} - \frac{16\sqrt{5\pi}(r-2M)(9M^2+19rM+11r^2)}{(2r+3M)^2r^6} \tag{63}$$

where the second line contains the "source" terms, corresponding to the quadratic contributions of the order J perturbations. To establish the relation between this function and the gravitational wave amplitude we consider the asymptotic behavior of the perturbations for large t, and r. To this end we expand the perturbations in sums of terms of the form $f_n(t-r^*)/r^n$, and obtain relations between the functions f_i, as in [10]. The results, restricted to leading orders in $1/r$, may be written in the form,

$$\begin{aligned}
H2_{2,0} &= 12\frac{G_0(t-r^*)}{r^3} + O(1/r^4) \\
h1_{2,0} &= -2\frac{G'_0(t-r^*)}{r} + O(1/r^2) \\
G_{2,0} &= \frac{G''_0(t-r^*)}{r} + 2\frac{G'_0(t-r^*)}{r^2} + O(1/r^3) \\
K_{2,0} &= 3\frac{G''_0(t-r^*)}{r} + 6\frac{G'_0(t-r^*)}{r^2} + O(1/r^3)
\end{aligned} \tag{64}$$

where $G_0(x)$ is a function of a single variable, determined by the initial data through the evolution equations, and a prime indicates a derivative. Replacing these expansions in (62), we find,

$$\chi^{J^2}_{2,0}(t,r) = G''_0(t-r^*) + \frac{3G'_0(t-r^*)}{r} + O(1/r^2) \tag{65}$$

Therefore, asymptotically, to leading order in $1/r$ we have $\chi^{J^2}_{2,0}(t,r) \simeq rG2_{2,0}(t,r)$, and $\chi^{J^2}_{2,0}$ may be identified directly with the gravitational wave amplitude. The initial data for solving (63) is obtained from the results of Section II. After replacement and simplification, we find,

$$\begin{aligned}
\chi^{J^2}_{2,0}|_{t=0} &= -\frac{4\sqrt{5\pi}(6r-5M)}{75r^2M(2r+3M)} \\
\left.\frac{\partial \chi_{2,0}}{\partial t}\right|_{t=0} &= 0
\end{aligned} \tag{66}$$

This is used in Section VI to obtain the indicated results.

C. Order $J\,L^2$, $\ell = 2$, even parity, perturbations

At order $J\,L^2$ we have $\ell = 2$, even, and $\ell = 3$, odd parity, contributions. Formally, these are of second order, since we are considering both J, and L^2 as first order, and they satisfy linear inhomogeneous equations, with source terms bilinear in the order J, and order L^2 terms. Now, given any solution of these equations, together with those of order J, and L^2, it is always possible to perform first a first order gauge transformation that takes the order J ($\ell = 1$, odd parity) terms to a standard form, that we have taken as the Kerr form, (number), and the order L^2 ($\ell = 2$, even parity) terms, to the standard Regge-Wheeler gauge, where one may write the Regge-Wheeler gauge functions uniquely in terms of the Zerilli function, ψ, (as in (52)), and the order L^2 Einstein equation are satisfied if ψ satisfies the Zerilli equation.

After the previous gauge transformations, a third gauge transformation may be used to put the $\ell = 2$, even parity, order JL^2 terms, also in a standard Regge - Wheeler gauge form. In this gauge we have that one of the Einstein equations takes the form,

$$H0_{2,2} = H2_{2,2} - \frac{8i}{3(r-2M)}\frac{\partial K^{L^2}_{2,2}}{\partial t} \tag{67}$$

where $K^{L^2}_{2,2}$ is of order L^2, and all other Regge-Wheeler coefficients are of order JL^2. Taking this into account, it is straightforward to prove that, as in the cases considered in [10], one can construct an infinite family of generalizations of the Zerilli function, that satisfy a Zerilli type inhomogeneous equation. From this family we have singled out the following function:

$$\chi_{2,2} = \frac{r(r-2M)}{3(2r+3M)}\left[H2_{2,2} - r\partial_r K_{2,2}\right] + \frac{1}{3}K_{2,2} - \frac{4i}{(2r+3M)}\partial_t \psi_{2,2} \tag{68}$$

It satisfies the equation,

$$\begin{aligned}\frac{\partial^2 \chi_{2,2}}{\partial t^2} &= \frac{\partial^2 \chi_{2,2}}{\partial r^{*2}} - 6\left(1-\frac{2M}{r}\right)\frac{(4r^3+4Mr^2+6rM^2+3M^3)}{r^3(2r+3M)^2}\chi_{2,2} \\ &\quad -\frac{16i(4r^3+56r^2M+36rM^2+15M^3)}{r^3(2r+3M)^3}\frac{\partial \psi_{2,2}}{\partial t}\end{aligned} \tag{69}$$

The corresponding inversion formulas are,

$$\begin{aligned}H2_{2,2} &= \frac{(2r^2-2rM+3M^2)}{r(2r+3M)}\frac{\partial \chi_{2,2}}{\partial r} - \frac{3(3M^3+6M^2r+4Mr^2+4r^3)}{r^2(2r+3M)^2}\chi_{2,2} \\ &\quad +(r-2M)\frac{\partial^2 \chi_{2,2}}{\partial r^2} - 4i\frac{(8r^4+124Mr^3+112M^2r^2+69M^3r+9M^4)}{r^2(r-2M)(2r+3M)^2)}\frac{\partial \psi_{2,2}}{\partial r} \\ &\quad -4i\frac{(2r^2-24rM-9M^2)}{3r(2r+3M)^2)}\frac{\partial^2 \psi_{2,2}}{\partial r \partial t} \\ H0_{2,2} &= H2_{2,2} - \frac{16i(r^2+rM+M^2)}{r^2(r-2M)(2r+3M)}\frac{\partial \psi_{2,2}}{\partial t} + \frac{8i}{3r}\frac{\partial^2 \psi_{2,2}}{\partial r \partial t} \\ H1_{2,2} &= \frac{(2r^2-6rM-3M^2)}{(r-2M)(2r+3M)}\frac{\partial \chi_{2,2}}{\partial t} + r\frac{\partial^2 \chi_{2,2}}{\partial r \partial t} \\ &\quad +\frac{4i(16r^5-192Mr^4-336M^2r^3-564M^3r^2-486M^4r-135M^5)}{r^4(2r+3M)^2}\psi_{2,2} \\ &\quad +\frac{4i(16r^3+28Mr^2+78M^2r+45M^3)}{3r^3(2r+3M)^2}\frac{\partial \psi_{2,2}}{\partial r} \\ &\quad +\frac{8i(r-2M)(9M^2+18Mr+r^2)}{3r^2(2r+3M)^2}\frac{\partial^2 \psi_{2,2}}{\partial r^2} \\ K_{2,2} &= \frac{6(r^2+rM+M^2)}{r^2(2r+3M)}\chi_{2,2} + \left(1-\frac{2M}{r}\right)\frac{\partial \chi_{2,2}}{\partial r} + \frac{4i(2r^2+16rM+9M^2)}{r^2(2r+3M)^2}\frac{\partial \psi_{2,2}}{\partial t}\end{aligned} \tag{70}$$

Again, to compute the asymptotic behavior of the Regge-Wheeler gauge functions. we need to find the asymptotic behavior of the corresponding Zerilli functions. Proceeding again as in [10], we obtain,

$$\chi_{2,2}(t,r) = \chi^{(0)}_{2,2}(t-r^*) + \chi^{(1)}_{2,2}(t-r^*)/r + \chi^{(2)}_{2,2}(t-r^*)/r^2 + \chi^{(3)}_{2,2}(t-r^*)/r^3 + \mathcal{O}(1/r^4) \tag{71}$$

where now we have,

$$\begin{aligned}\chi^{(0)}_{2,2}(x) &= \frac{d^2\mathcal{G}_{2,2}}{dx^2} \\ \chi^{(1)}_{2,2}(x) &= 3\frac{d\mathcal{G}_{2,2}}{dx} \\ \chi^{(2)}_{2,2}(x) &= 3\mathcal{G}_{2,2} - 3M\frac{d\mathcal{G}_{2,2}}{dx} + 2i\frac{d^2\mathcal{F}_{2,2}}{dx^2} \\ \chi^{(3)}_{2,2}(x) &= -3M\mathcal{G}_{2,2} + \frac{21M^2}{4}\frac{d\mathcal{G}_{2,2}}{dx} + 4i\frac{d\mathcal{F}_{2,2}}{dx} + \frac{46iM}{3}\frac{d^2\mathcal{F}_{2,2}}{dx^2}\end{aligned} \tag{72}$$

and similar expressions for $\chi_{2,-2}$. The functions $\mathcal{G}_{2,2}$ are uniquely determined by the initial data. This is obtained starting with the Bowen - York initial data, and first performing a gauge transformation of the order J, $\ell = 1$, odd parity perturbations to a Boyer-Lindquist-Kerr gauge on the whole initial data, up to order JL^2. This does not modify the order L^2 terms but induces a (linear in J) change in the order JL^2 terms, and a second order gauge transformation on the order J^2 terms. On this modified data we perform a second gauge transformation, this time on the order L^2, $\ell = 2$, even parity perturbations to carry it to the Regge - Wheeler gauge. This induces a further (linear) transformation on the order JL^2 perturbations. Actually, we do not need this last transformation because the homogeneous part of (68), (the part independent of $\psi_{2,2}$) is gauge invariant under gauge transformations of order JL^2 of the $\ell = 2$, even parity perturbations, and, therefore, we may obtain that part of the initial data without explicitly going to the Regge - Wheeler gauge. After some simplifications, the initial data for $\chi_{2,2}(t,r)$ may be written in form,

$$
\begin{aligned}
\chi_{2,2}|_{t=0} &= 0 \\
\left.\frac{\partial \chi_{2,2}}{\partial t}\right|_{t=0} &= -\frac{i\sqrt{30\pi}\rho^{1/2}}{r^{5/2}(2r+3M)^2}\left[\frac{28}{45}+\frac{17}{15}\frac{M}{\rho}\right. \\
&\quad \left. -\frac{89}{180}\frac{M^2}{\rho^2}+\frac{21}{20}\frac{M^3}{\rho^3}+\frac{119}{360}\frac{M^4}{\rho^4}+\frac{1}{96}\frac{M^5}{\rho^5}\right]
\end{aligned} \tag{73}
$$

where $\rho = (\sqrt{r}+\sqrt{r-2M})^2/4$.

We remark that the terms with $\ell = 2$, $m = 0$, even parity, of order $J\,L^2$ decouple from the order J perturbations, and satisfy homogeneous equations. Since the initial data for these terms vanish, they vanish and make no contributions to the evolution of the system.

For the computation of the radiated energy and wave forms, we need the asymptotic behavior of the perturbations in an asymptotically flat gauge. It can be checked that in such a gauge, for large r, and t, we have

$$\chi_{2,-2}(t,r) \simeq r G_{2,-2}(t,r) \tag{74}$$

where $\simeq$ implies equality up to terms of order r^{-1}, and therefore $\chi_{2,-2}(t,r)$ is directly related to the gravitational wave amplitude. This result is used in Sections XX and XX, to compute the radiation of energy and angular momentum.

D. Order JL^2, $\ell = 3$, $m = \pm 2$, odd parity, perturbations

In this case we may consider again transformations that carry the metric to the Regge-Wheeler gauge form. The order JL^2, $\ell = 3$, odd parity, perturbations satisfy linear equations with a "source term" that is linear in the order L^2, $\ell = 2$, even parity, perturbations. In the Regge-Wheeler gauge these may be expressed entirely in terms of the corresponding Zerilli function. One may in this case verify that if the Einstein equations are satisfied, then the Regge-Wheeler function defined by,

$$Q_{3,2}(t,r) = -\frac{r}{10}\left[\frac{2}{r}k0_{3,2} - \frac{\partial k0_{3,2}}{\partial r} + \frac{\partial k1_{3,2}}{\partial t}\right] - \frac{\sqrt{7}}{105}\left[\frac{\partial K^{L^2}_{2,2}}{\partial r} + \frac{3}{r}K^{L^2}_{2,2}\right] \tag{75}$$

where $K^{L^2}_{2,2}$ is of order L^2, and all other Regge-Wheeler coefficients are of order JL^2, satisfies the equation,

$$\frac{\partial^2 Q_{3,2}}{\partial t^2} - \frac{\partial^2 Q_{3,2}}{\partial {r^*}^2} + \frac{6(r-2M)(2r-M)}{r^4}Q_{3,2} + \frac{4(r-2M)}{\sqrt{7}r^4}K^{L^2}_{2,2} = 0 \tag{76}$$

where

$$K^{L^2}_{2,2} = \frac{6(r^2+rM+M^2)}{r^2(2r+3M)}\psi_{2,2} + \left(1-\frac{2M}{r}\right)\frac{\partial \psi_{2,2}}{\partial r} \tag{77}$$

Using these, and the Einstein equations, one finds the following "inversion" formulas,

$$
\begin{aligned}
k0_{3,2} &= (r-2M)\frac{\partial Q_{3,2}}{\partial r} + \frac{(r-2M)}{r}Q_{3,2} + \frac{2}{3\sqrt{7}r}K^{L^2}_{2,2} \\
k1_{3,2} &= \frac{r^2}{(r-2M)}\frac{\partial Q_{3,2}}{\partial t}
\end{aligned} \tag{78}
$$

The asymptotic behavior of $Q_{3,2}$ for large r, and t, is easily obtained from (76). It can written in the form,

$$Q_{3,2} \simeq \mathcal{Q}_0(t-r^*) + \mathcal{Q}_1(t-r^*)\frac{1}{r}\mathcal{Q}_2(t-r^*)\frac{1}{r^2} + \mathcal{O}(1/r^3) \tag{79}$$

where

$$\begin{aligned}
\mathcal{Q}_0(t) &= \frac{d^2\mathcal{Q}(t)}{dt^2} \\
\mathcal{Q}_1(t) &= 6\frac{d\mathcal{Q}(t)}{dt} \\
\mathcal{Q}_2(t) &= 15\mathcal{Q}(t) - \frac{3M}{2}\frac{d\mathcal{Q}(t)}{dt} - \frac{\sqrt{7}}{21}\frac{\partial^2\mathcal{K}(t)}{\partial t^2}
\end{aligned} \tag{80}$$

with $\mathcal{Q}$ a certain function that depends on the initial data, and $\mathcal{K}$ related to $\psi_{2,2}$ by, $\partial^2\mathcal{K}(t-r^*)/\partial t^2 \simeq \psi_{2,2}(t,r)$, for large r, and t.

Using these results, and the gauge transformation that relates the Regge-Wheeler gauge to an asymptotically flat gauge we find that for large r, and t, we have,

$$k2_{3,2}(t,r) \simeq -2rQ_{3,2}(t,r) \simeq -2r\mathcal{Q}_0(t-r^*) \tag{81}$$

The solution for $m=-2$ is the complex conjugate of that for $m=2$. Therefore, solving (76 we find the gravitational wave amplitude for this mode.

The foregoing construction gives the complete solution of the perturbative Einstein equations for the mode considered. For computational purposes, however, if one is interested only in finding the radiated energy and wave forms, it is simpler to notice that, after fixing the gauge for the order J ("Kerr" gauge), and for the order L^2 (Regge-Whheler gauge) perturbations, in a general gauge for the order JL^2 perturbations, the function,

$$\widetilde{Q}_{3,2}(t,r) = \frac{(r-2M)}{r^2}\left[\widetilde{k1}_{3,2} + \frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{\widetilde{k2}_{3,2}}{r^2}\right)\right] \tag{82}$$

satisfies the inhomogeneous Regge-Wheeler equation,

$$\begin{aligned}
\frac{\partial^2\widetilde{Q}_{3,2}}{\partial t^2} &= \frac{\partial^2\widetilde{Q}_{3,2}}{\partial {r^*}^2} - \frac{6(r-2M)(2r-M)}{r^4}\widetilde{Q}_{3,2} \\
&\quad -\frac{4(r-2M)^2}{\sqrt{7}r^5}\left[\frac{\partial^2\psi_{2,2}}{\partial t\partial r} + \frac{6(r^2+rM+M^2)}{r(r-2M)(2r+3M)}\frac{\partial\psi_{2,2}}{\partial t}\right]
\end{aligned} \tag{83}$$

Asymptotically for large r, and t, $\widetilde{Q}_{3,2}$ admits the expansion,

$$\widetilde{Q}_{3,2}(t,r) = \widetilde{\mathcal{Q}}_0(t-r^*) + \mathcal{O}(r^{-1}) \tag{84}$$

We remark that, after fixing the gauges for the lower order perturbations, $\widetilde{Q}_{3,2}$ is gauge invariant under transformations of order JL^2. In particular, in an asymptotically flat gauge for these perturbations we have,

$$\widetilde{k2}_{3,2}(t,r)_{,r} \simeq 2r\widetilde{Q}_{3,2}(t,r) \simeq -2r\widetilde{\mathcal{Q}}_0(t-r^*) \tag{85}$$

and, therefore,

$$\widetilde{k2}_{3,2}(t,r)_{,t} \simeq -2r\widetilde{Q}_{3,2}(t,r) \tag{86}$$

where $\simeq$ implies equality up to terms that decrease as $1/r$ for large r. This last equation establishes the relation between $\widetilde{Q}_{3,2}$ and the gravitational wave amplitude.

To obtain the initial data for $\widetilde{Q}_{3,2}$, we start with the Bowen-York initial data, and perform firt a first order transformation on the order J, $\ell=1$, odd parity perturbations, that take these to the "Kerr" form, and then, on the result, we perform an order L^2 transformation on the $\ell=2$, even parity, to carry these to the Regge-Wheeler gauge. Using the resulting form for the order JL^2, $\ell=3$, odd parity perturbations, we find that the initial data for $\widetilde{Q}_{3,2}$ may be written in the form

$$\begin{aligned}
\widetilde{Q}_{3,2}(t,r)|_{t=0} &= 0 \\
\left.\frac{\partial\widetilde{Q}_{3,2}}{\partial t}\right|_{t=0} &= \frac{2\sqrt{210\pi}\,M\,\sqrt{r-2\,M}}{21(\sqrt{r}+\sqrt{r-2\,M})^6\,r^{(9/2)}}\left[5\left(\sqrt{r}+\sqrt{r-2\,M}\right)^2 - 6M\right]
\end{aligned} \tag{87}$$

and we have the same initial data for $m=-2$.

E. Order JL^2, $\ell = 3$, $m = 0$, odd parity, perturbations

In this case the Regge-Wheeler function is given by,

$$Q_{3,0}(t,r) = -\frac{r}{10}\left[\frac{2}{r}k0_{3,0} - \frac{\partial k0_{3,0}}{\partial r} + \frac{\partial k1_{3,0}}{\partial t}\right] - \frac{\sqrt{35}}{175}\left[\frac{\partial K_{2,0}}{\partial r} + \frac{3}{r}K^{L^2}_{2,0}\right] \tag{88}$$

where again $K^{L^2}_{2,2}$ is of order L^2, and all other Regge-Wheeler coefficients are of order JL^2, and the Regge-Wheeler equation takes the form,

$$\frac{\partial^2 Q_{3,0}}{\partial t^2} - \frac{\partial^2 Q_{3,0}}{\partial r^{*2}} + \frac{6(r-2M)(2r-M)}{r^4}Q_{3,0} + \frac{12(r-2M)}{\sqrt{35}r^4}K^{L^2}_{2,0} = 0 \tag{89}$$

where

$$K^{L^2}_{2,0} = \frac{6(r^2+rM+M^2)}{r^2(2r+3M)}\psi_{2,0} + \left(1 - \frac{2M}{r}\right)\frac{\partial \psi_{2,0}}{\partial r} \tag{90}$$

and the "inversion" formulas are,

$$\begin{aligned} k0_{3,0} &= (r-2M)\frac{\partial Q_{3,0}}{\partial r} + \frac{(r-2M)}{r}Q_{3,0} + \frac{2}{\sqrt{35}r}K^{L^2}_{2,0} \\ k1_{3,0} &= \frac{r^2}{(r-2M)}\frac{\partial Q_{3,0}}{\partial t} \end{aligned} \tag{91}$$

As in the case $m = \pm 2$, the asymptotic behavior of $Q_{3,0}$ for large r, and t, is easily obtained from (89). It can written in the form,

$$Q_{3,0} \simeq \mathcal{Q}_0(t-r^*) + \mathcal{Q}_1(t-r^*)\frac{1}{r}\mathcal{Q}_2(t-r^*)\frac{1}{r^2} + \mathcal{O}(1/r^3) \tag{92}$$

where

$$\begin{aligned} \mathcal{Q}_0(t) &= \frac{d^2\mathcal{Q}(t)}{dt^2} \\ \mathcal{Q}_1(t) &= 6\frac{d\mathcal{Q}(t)}{dt} \\ \mathcal{Q}_2(t) &= 15\mathcal{Q}(t) - \frac{3M}{2}\frac{d\mathcal{Q}(t)}{dt} - \frac{1}{\sqrt{35}}\frac{\partial^2\mathcal{K}(t)}{\partial t^2} \end{aligned} \tag{93}$$

with $\mathcal{Q}$ again a certain function that depends on the initial data, and $\mathcal{K}$ related to $\psi_{2,0}$ by, $\partial^2\mathcal{K}(t-r^*)/\partial t^2 \simeq \psi_{2,0}(t,r)$, for large r, and t.

Just as in the case $m = \pm 2$,we find that for large r, and t, the solution of (89) is related to the amplitude in an asymptotically flat gauge by,

$$k2_{3,0}(t,r) \simeq -2rQ_{3,0}(t,r) \simeq -2r\mathcal{Q}_0(t-r^*) \tag{94}$$

Thus, again the solution of (89) provides the gravitational wave amplitude for this mode.

As in the previous subsection, if one is interested only in finding the radiated energy and wave forms, we notice that, after fixing the gauge for the order J ("Kerr" gauge), and for the order L^2 (Regge-Wheeler gauge) perturbations, in a general gauge for the order JL^2 perturbations, the function,

$$\widetilde{Q}_{3,0}(t,r) = \frac{(r-2M)}{r^2}\left[\widetilde{k1}_{3,0} + \frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{\widetilde{k2}_{3,0}}{r^2}\right)\right] \tag{95}$$

satisfies the equation,

$$\begin{aligned} \frac{\partial^2 \widetilde{Q}_{3,0}}{\partial t^2} &= \frac{\partial^2 \widetilde{Q}_{3,0}}{\partial r^{*2}} - \frac{6(r-2M)(2r-M)}{r^4}\widetilde{Q}_{3,0} \\ &\quad - \frac{12(r-2M)^2}{\sqrt{35}r^5}\left[\frac{\partial^2\psi_{2,0}}{\partial t\partial r} + \frac{6(r^2+rM+M^2)}{r(r-2M)(2r+3M)}\frac{\partial \psi_{2,0}}{\partial t}\right] \end{aligned} \tag{96}$$

Asymptotically for large r, and t, $\widetilde{Q}_{3,0}$ admits the expansion,

$$\widetilde{Q}_{3,0}(t,r) = \widetilde{\mathcal{Q}}_0(t-r^*) + \mathcal{O}(r^{-1}) \tag{97}$$

We also have in this case that, after fixing the gauges for the lower order perturbations, $\widetilde{Q}_{3,0}$ is gauge invariant under transformations of order JL^2. In an asymptotically flat gauge for these perturbations we have,

$$\widetilde{k2}_{3,0}(t,r)_{,r} \simeq 2r\widetilde{Q}_{3,0}(t,r) \simeq -2r\widetilde{\mathcal{Q}}_0(t-r^*) \tag{98}$$

and, therefore,

$$\widetilde{k2}_{3,0}(t,r)_{,t} \simeq -2r\widetilde{Q}_{3,0}(t,r) \tag{99}$$

which provides the relation between $\widetilde{Q}_{3,0}$ and the gravitational wave amplitude.

Proceeding as in the case $m=\pm 2$, the initial data for $\widetilde{Q}_{3,0}$ may be written in the form

$$\begin{aligned} \widetilde{Q}_{3,0}(t,r)|_{t=0} &= 0 \\ \left.\frac{\partial \widetilde{Q}_{3,0}}{\partial t}\right|_{t=0} &= -\frac{4\sqrt{7\pi}\,M\,\sqrt{r-2\,M}}{7(\sqrt{r}+\sqrt{r-2\,M})^6\,r^{(9/2)}}\left[5\left(\sqrt{r}+\sqrt{r-2\,M}\right)^2-6M\right] \end{aligned} \tag{100}$$

V. EVOLUTION AND WAVEFORMS

All the relevant Regge-Wheeler and Zerilli equations were evolved using straightforward modifications of the routines used in [18]. As indicated there, the evolution is carried out in the Cauchy domain of some appropriately chosen interval of the r^*-axis, and therefore, we do not need to impose any boundary conditions. To obtain a definite waveform we consider an extraction point at r of the order of $100M$, (the exact point is actually irrelevant, since the amplitudes are independent of r, for large r), and register the different amplitudes as functions of integration time. For the $\ell = 2$, orders J^2 and L^2, even parity terms we obtain the familiar "quasinormal ringing" waveforms. This is also true for the order JL^2, odd parity, $\ell = 3$, amplitudes, although the ringdown frequencies are different from those for $\ell = 2$. For the $\ell = 2$, even parity perturbations of order JL^2, we obtain a waveform of the same frequency as that for $\ell = 2$ of order L^2, but with a different initial shape, and a noticeable shift in phase. This last feature is essential for the radiation of angular momentum. As illustrations of these results we include in Figure 1 a comparison of the wave forms discussed above. Notice that for χ we have indicated the imaginary part (the real part vanishes in this case)

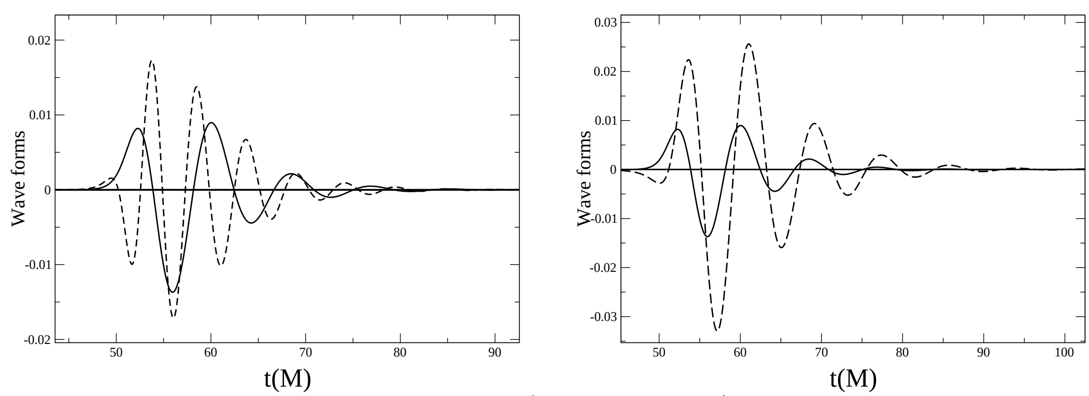

FIG. 1. In the figure on the left the solid curve represents (in arbitrary units) the wave form corresponding to $\psi_{2,2}$, and the dashed curve that of $Q_{3,2}$, as functions of time, for fixed r. Notice the difference in ring down frequency, for the different values of ℓ. The figure on the right compares, again in arbitrary units, the wave forms corresponding to $\psi_{2,2}$ (solid curve), and to $\chi_{2,2}$ (dashed curve). Here we notice the large difference in phase, giving rise to radiation of angular momentum.

VI. RADIATED ENERGY

To compute the power and radiated energy we consider the metric written in a transverse traceless, asymptotically flat gauge. These conditions are satisfied asymptotically in what we have called a "Misner" gauge in this paper. We may therefore apply the formula [19]

$$\frac{d\mathrm{Power}}{d\Omega} = \lim_{r\to\infty} \frac{1}{16\pi r^2}\left[\left(\frac{1}{\sin\theta}\frac{\partial h_{\theta\phi}}{\partial t}\right)^2 + \frac{1}{4}\left(\frac{\partial h_{\theta\theta}}{\partial t} - \frac{1}{\sin^2\theta}\frac{\partial h_{\phi\phi}}{\partial t}\right)^2\right] \tag{101}$$

Using the general form of the Regge - Wheeler expansion given in [10], for large r, we have,

$$\begin{aligned}
h_{\theta\theta} - \frac{1}{\sin^2\theta}h_{\phi\phi} &= r^2\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{m=\ell} G_{\ell,m}\left[\frac{\partial^2\, Y_\ell{}^m}{\partial\theta^2} - \frac{1}{\sin^2\theta}\frac{\partial^2 Y_\ell{}^m}{\partial\phi^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial\, Y_\ell{}^m}{\partial\theta}\right]\\
&\quad +2\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{m=\ell} k2_{\ell,m}\left[\frac{1}{\sin(\theta)}\frac{\partial^2 Y_\ell{}^m}{\partial\theta\,\partial\phi} - \frac{\cos(\theta)}{\sin^2(\theta)}\frac{\partial Y_\ell{}^m}{\partial\phi}\right]\\
\frac{1}{\sin\theta}h_{\theta\phi} &= \frac{1}{2}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{m=\ell} k2_{\ell,m}\left[\frac{\partial^2\, Y_\ell{}^m}{\sin^2\theta\partial\phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial\, Y_\ell{}^m}{\partial\theta} - \frac{\partial^2\, Y_\ell{}^m}{\partial\theta^2}\right]\\
&\quad +r^2\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{m=\ell} G_{\ell,m}\left[\frac{\partial^2 Y_\ell{}^m}{\sin\theta\partial\theta\,\partial\phi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial Y_\ell{}^m}{\partial\phi}\right]
\end{aligned} \tag{102}$$

where the sums over ℓ, and m are restricted to the those of the perturbation considered. After expansion of the coefficients in the form,

$$\begin{aligned}
G_{\ell,m}(t,r) &= J^2 G^{(J^2)}_{\ell,m}(t,r) + L^2 G^{(L^2)}_{\ell,m}(t,r) + JL^2 G^{(JL^2)}_{\ell,m}(t,r)\\
k2_{\ell,m}(t,r) &= JL^2 k2^{(JL^2)}_{\ell,m}(t,r)
\end{aligned} \tag{103}$$

where we have explicitly indicated the orders of the terms, an integration over the angles gives,

$$\begin{aligned}
\mathrm{Power} = \lim_{r\to\infty}\frac{3r^2}{8\pi}&\left[J^4\left(\dot G^{(J^2)}_{2,0}\right)^2 + 2J^2L^2\dot G^{(J^2)}_{2,0}\dot G^{(L^2)}_{2,0} + L^4\left(\left(\dot G^{(L^2)}_{2,0}\right)^2 + 2\dot G^{(L^2)}_{2,2}\dot G^{(L^2)}_{2,-2}\right)\right.\\
&\left. + \frac{5J^2L^4}{r^4}\left(\left(\dot k2^{(JL^2)}_{2,0}\right)^2 + 2\dot k2^{(JL^2)}_{2,2}\dot k2^{(JL^2)}_{2,-2}\right) + 2J^2L^4\dot G^{(JL^2)}_{2,2}\dot G^{(JL^2)}_{2,-2}\right]
\end{aligned} \tag{104}$$

where an overdot indicates $\partial/\partial t$. Therefore, in terms of the corresponding Regge - Wheeler and Zerilli functions, the total radiated energy is given by,

$$\begin{aligned}
\mathrm{Energy} = \lim_{r\to\infty}\frac{3}{8\pi}\int_0^\infty &\left[J^4\left(\dot\chi^{(J^2)}_{2,0}\right)^2 + 2J^2L^2\dot\chi^{(J^2)}_{2,0}\dot\psi^{(L^2)}_{2,0} + 4L^4\left(\dot\psi^{(L^2)}_{2,0}\right)^2\right.\\
&\left. +20J^2L^4\left(\left(\dot Q^{(JL^2)}_{2,0}\right)^2 + 2\dot Q^{(JL^2)}_{2,2}\dot Q^{(JL^2)}_{2,-2}\right) + 2J^2L^4\dot\chi^{(JL^2)}_{2,2}\dot\chi^{(JL^2)}_{2,-2}\right] dt
\end{aligned} \tag{105}$$

After numerical integration of the evolution equations, with the initial data derived previously, we obtain

$$\begin{aligned}
\mathrm{Energy}/M &= 7.8\times10^{-4}(J/M^2)^4 - 1.4\times10^{-5}(J/M^2)^2(L/M)^2 + 2.45\times10^{-5}(L/M)^4\\
&\quad +7.63\times10^{-4}(J/M^2)^2(L/M)^4(\ell=2) + 0.96\times10^{-4}(J/M^2)^2(L/M)^4(\ell=3)
\end{aligned} \tag{106}$$

where in the second line we have written separately the $\ell = 2$, even parity, and $\ell = 3$, odd parity contributions of order JL^2.

VII. RADIATION OF ANGULAR MOMENTUM

The lowest order in the radiation of angular momentum comes from the interference of the $\ell = 2$, $m = \pm 2$, even parity, outgoing waves of order L^2, with those of order JL^2, giving a contribution of order JL^4. We may obtain the expression for the radiated angular momentum using the formulas in [20]. This implies, however, that we need to obtain the asymptotic transverse traceless gauge amplitudes, which implies a rather lengthy computation. We may instead use the fact that the following expression,

$$\mathcal{J} = \lim_{r\to\infty} \left[\frac{1}{4\sqrt{3\pi}} \left(r^2 k0_{1,0},_r(r,t) - 2rk0_{1,0}(r,t)r^2 - k1_{1,0},_t(r,t) \right) \right] \tag{107}$$

where $k0_{1,0}(r,t)$, and $k1_{1,0}(r,t)$ represent the corresponding Regge-Wheeler coefficients in an asymptotic expansion of the metric for large r, corresponds, for $t = 0$, and to leading order, to the (ADM) angular momentum on the initial hypersurface. It is clear that $\mathcal{J}$, defined by (107) is time dependent. Since $\mathcal{J}(t = 0)$ represents the initial value of the angular momentum of the system, we should consider $\lim_{t\to\infty} \mathcal{J}(t)$, as the final value of the angular momentum, after the system has settled to its final Kerr black hole. The radiated angular momentum is therefore,

$$\Delta\mathcal{J} = \mathcal{J}(0) - \lim_{t\to\infty} \mathcal{J}(t) \tag{108}$$

To compute $\Delta\mathcal{J}$ perturbatively, we expand $k0_{1,0}(r,t)$, $k1_{1,0}(r,t)$, in powers of the perturbation parameters J, and L,

$$\begin{aligned} k0_{1,0}(r,t) &= J\, k0^{(1)}_{1,0}(r,t) + JL^2\, k0^{(2)}_{1,0}(r,t) + JL^4\, k0^{(3)}_{1,0}(r,t) + J^2L^4\, k0^{(4)}_{1,0}(r,t) + \ldots \\ k1_{1,0}(r,t) &= J\, k1^{(1)}_{1,0}(r,t) + JL^2\, k1^{(2)}_{1,0}(r,t) + JL^4\, k1^{(3)}_{1,0}(r,t) + J^2L^4\, k1^{(4)}_{1,0}(r,t) + \ldots \end{aligned} \tag{109}$$

Notice that there are no contributions of order J^2, L^2, and L^4. If we consider now the Einstein equations for the different orders, we have that, to order J,

$$\mathcal{J}^{(1)} = \left[\frac{1}{4\sqrt{3\pi}} \left(r^2\, k0^{(1)},_r(r,t) - 2rk0^{(1)}(r,t) - r^2 k1^{(1)},_t(r,t) \right) \right] \tag{110}$$

is a gauge invariant constant, that, on account of the form of the initial data, is equal to $\mathcal{J}$ on the initial hypersurface.

For the higher order terms, a simple, but rather lengthy calculation shows that, we have,

$$\frac{\partial}{\partial t} \left[r^2 k0^{(i)},_r(r,t) - 2rk0^{(i)}(r,t) - r^2 k1^{(i)},_t(r,t) \right] = \mathcal{S}^{(i)} \tag{111}$$

where $\mathcal{S}^{(i)}$ is a 'source', that depends on the lower order perturbations. We then have,

$$\Delta\mathcal{J} = -\lim_{r\to\infty} \int_0^\infty \left[\mathcal{S}^{(2)} + \mathcal{S}^{(3)} + \mathcal{S}^{(4)} + \ldots \right] dt \tag{112}$$

One can show that the net contribution from $\mathcal{S}^{(2)}$ vanishes. Then, the leading contribution to $\Delta\mathcal{J}$ comes from $\mathcal{S}^{(3)}$. From Einstein's equations we find that $\mathcal{S}^{(3)}$ is bilinear in the $\ell = 2$,$m = \pm 2$, order L^2 and order JL^2 contributions. Writing these perturbations in the Regge - Wheeler gauge, and using the expansions of the previous Section, a straightforward computation shows that,

$$\Delta\mathcal{J} = \frac{3i}{4\pi} \lim_{r\to\infty} \int_0^\infty \left[\frac{\partial\chi_{2,2}}{\partial t}\psi_{2,-2} - \frac{\partial\psi_{2,-2}}{\partial t}\chi_{2,2} - \frac{\partial\chi_{2,-2}}{\partial t}\psi_{2,2} + \frac{\partial\psi_{2,2}}{\partial t}\chi_{2,-2} \right] dt \tag{113}$$

Using now the results of the numerical evolution, we finally find,

$$\frac{\Delta\mathcal{J}}{\mathcal{J}} = 0.00088 \left(\frac{L}{M} \right)^4 \tag{114}$$

This result implies that less than 0.1% of the total angular momentum will radiated even in the extreme case $L \simeq M$. It is in good agreement with the corresponding result found in [11] for the "pseudo-inspiral" case.

VIII. FINAL COMMENTS AND CONCLUSIONS

In this paper we have numerically evolved a particular set of initial data, and obtained quantitative expressions for the radiated energy and angular momentum, for a system of two equal mass rotating black holes, in the close limit. If we analyze in more detail the expression for the radiated energy, we find that a large contribution comes from the order J^2 amplitude. This is probably due to the fact that the initial data for rotating black holes obtained with the Bowen and York prescription, even for a single black hole [15], contains a fair amount of "spurious" radiation that must be radiated away before the system can settle to the final Kerr form. One might then expect that they would be absent, or make a much less significant contribution if the evolution started with initial data closer to that of a Kerr black hole. On the other hand, it seems reasonable to assume that the contributions of the terms of order L^2, and JL^2 should be much less dependent on the form of the initial data, since for order L^2 they are independent of spin, and for order JL^2 they require the order J data, and this is "exactly" of the Kerr form. It is precisely these terms that provide the leading contribution to the radiated angular momentum. This contribution is such as to decrease the angular momentum of the system. The radiation may therefore be considered as providing a "torque" somewhat in the manner envisioned in [4].

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